MATB44 Week 11 Notes

1. Series Solns Near An Ordinary Point:

E.g. 1 Determine a series soln for the following diff eqn about $x_0 = 0$.

$$y'' + y = 0$$

Soln:

Recall the power series summation is $\sum_{n=0}^{\infty} a_n (x-x_0)^n$.

Let $y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=1}^{\infty} a_n (n) x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

We want both y and y'' to have x^n.
Hence, we'll have to change y''.

$$y'' = \sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n$$

Recall: If you add s to the starting value, you subtract s from the formula and if you subtract s from the starting value, you add s to the function.

Now, $y'' + \underset{n}{y} \overset{=0}{}$ looks like

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Combining the 2 summations together, we get

$$\sum_{n=0}^{\infty} x^n \left[a_{n+2}(n+2)(n+1) + a_n\right] = 0$$

This means that $a_{n+2} = \dfrac{-a_n}{(n+2)(n+1)}$, $n = 0, 1, 2, \ldots$

Now, let's plug some values for n.

$n=0$:

$a_2 = \dfrac{-a_0}{2\cdot 1}$

$n=1$:

$a_3 = \dfrac{-a_1}{3\cdot 2}$

$n=2$:

$a_4 = \dfrac{-a_2}{4\cdot 3}$

$= \dfrac{a_0}{4\cdot 3\cdot 2\cdot 1}$

$n = 3$:

$$a_5 = \frac{-a_3}{5 \cdot 4}$$

$$= \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}$$

We begin to see a pattern:

$$a_{2k} = \frac{(-1)^k a_0}{(2k)!} \quad \text{and} \quad a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!}$$

Recall that $y = \sum_{n=0}^{\infty} a_n x^n$.

Expanding the summation, we get

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= a_0 + a_1 x - \frac{a_0 x^2}{2} - \frac{a_1 x^3}{6} + \ldots$$

$$= a_0 \left(1 - \frac{x^2}{2!} + \ldots \right) + a_1 \left(x - \frac{x^3}{3!} + \ldots \right)$$

$$= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

E.g. 2 Determine a series soln for $y''-y=0$ about $x_0=0$.

Soln:

$$y=\sum_{n=0}^{\infty} a_n x^n, \quad y'=\sum_{n=1}^{\infty} a_n (n) x^{n-1}, \quad y''=\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

$y''-y=0$ can be rewritten as

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} x^n \left[a_{n+2}(n+2)(n+1) - a_n\right] = 0$$

Note: For this eqn to be satisfied for all x, the coefficient of each power of x must be 0. Hence, we get $\underbrace{a_{n+2}(n+2)(n+1) - a_n = 0 \quad n=0,1,2,\ldots}_{\text{Recurrence Relation}}$

$$a_{n+2}(n+2)(n+1) - a_n = 0$$

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)}$$

Now, let's plug some values for n.

$n=0$:

$$a_2 = \frac{a_0}{(2)(1)}$$

$n=1$:

$$a_3 = \frac{a_1}{3 \cdot 2}$$

$n=2$:

$$a_4 = \frac{a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{a_0}{4!}$$

$n=3$:

$$a_5 = \frac{a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2} = \frac{a_1}{5!}$$

$n=4$:

$$a_6 = \frac{a_4}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 4!} = \frac{a_0}{6!}$$

$n=5$:

$$a_7 = \frac{a_5}{7 \cdot 6} = \frac{a_1}{7 \cdot 6 \cdot 5!} = \frac{a_1}{7!}$$

We begin to see a pattern.

$$a_{2k} = \frac{a_0}{(2k)!}, \quad k = 1, 2, \ldots$$

$$a_{2k+1} = \frac{a_1}{(2k+1)!}, \quad k = 1, 2, \ldots$$

Recall that $y = \sum_{n=0}^{\infty} a_n x^n$.

Expanding the summation, we get:

$$\begin{aligned} y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \\ &= a_0 + a_1 x + \frac{a_0 x^2}{2!} + \frac{a_1 x^3}{3!} \\ &= a_0 \left(1 + \frac{x^2}{2!} + \ldots\right) + a_1 \left(x + \frac{x^3}{3!} + \ldots\right) \\ &= a_0 \underbrace{\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}}_{y_1} + a_1 \underbrace{\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}}_{y_2} \end{aligned}$$

E.g. 3 Find a series soln for $y'' + 3y' = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} a_n (n) x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

We want both y' and y'' to have x^n. Hence, we have to rewrite the summations.

$$y' = \sum_{n=0}^{\infty} a_{n+1} (n+1) x^n, \quad y'' = \sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n$$

We can rewrite $y'' + 3y' = 0$ as

$$\sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n + \sum_{n=0}^{\infty} 3a_{n+1} (n+1) x^n = 0$$

Combining the summations, we get

$$\sum_{n=0}^{\infty} x^n \left[a_{n+2} (n+2)(n+1) + 3a_{n+1} (n+1) \right] = 0$$

Hence, we get

$$a_{n+2} (n+2)(n+1) + 3a_{n+1} (n+1) = 0$$

$$a_{n+2} = \frac{-3a_{n+1}}{n+2}$$

Now, we plug values in for n.

$n=0$:

$$a_2 = \frac{-3a_1}{2}$$

$n=1$:

$$a_3 = \frac{-3a_2}{3} = \frac{9a_1}{3\cdot 2} = \frac{9a_1}{3!}$$

$n=2$:

$$a_4 = \frac{-3a_3}{4} = \frac{-3(\overset{9}{}a_1)}{4\cdot 3\cdot 2} = \frac{-27a_1}{4!}$$

$n=3$:

$$a_5 = \frac{-3a_4}{5} = \frac{81a_1}{5!}$$

We can see a pattern.

$$a_k = \frac{(-3)^k a_1}{(k+1)!}, \quad k = 1, 2, \ldots$$

$y_1 = 1$ ← Because we don't have a_0.

$$y_2 = a_1 \sum_{n=0}^{\infty} \frac{(-3)^n x^{n+1}}{(n+1)!}$$

$$y = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= \underbrace{1}_{y_1} + \underbrace{a_1 x + \frac{(-3)a_1 x^2}{2!} + \frac{(-3)^2 a_1 x^3}{3!} + \ldots}_{y_2}$$

2. Series Soln Near A Regular Singular Point:

- Consider $P(t)y'' + Q(t)y' + R(t)y = 0$.
t_0 is called a **singular point** if $P(t_0) = 0$.
- When we want to find a series soln and we have a singular point, we let

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

- **Note:** By convention, we assume that $a_0 \neq 0$.

Here's what would happen if $a_0 = 0$:

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

$$= \underbrace{x^r a_0}_{=0} + x^r \sum_{n=1}^{\infty} a_n x^n$$

$$= x^r \sum_{n=1}^{\infty} a_n x^n$$

$$= x^r \sum_{n=0}^{\infty} a_{n+1} x^{n+1}$$

$$= x^{r+1} \sum_{n=0}^{\infty} a_{n+1} x^n$$

E.g. 4 Find a series soln for
$2x^2y'' - xy' + (1+x)y = 0$ about $x_0 = 0$.

Soln:
First, notice that $x_0 = 0$ is a singular point.
$P = 2x^2 \rightarrow P(0) = 0$
Hence, we use

$$y = x^r \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$= \sum_{n=0}^{\infty} a_n x^{n+r} \qquad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

Now, we can rewrite $2x^2y'' - xy' + \overbrace{y + xy}^{\text{Same as } (1+x)y} = 0$ as

$$2x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} -$$

$$x \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} + x \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\rightarrow 2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

We want all the summations to have x^{n+r}, so we have to change the last summation.

$$\sum_{n=0}^{\infty} a_n x^{n+r+1} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Now, we have

$$\sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)x^{n+r} - a_n(n+r)x^{n+r} + a_n x^{n+r} +$$

$$\sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0$$

Take $n=0$. We get:

$2a_0(r)(r-1) - a_0(r) + a_0 = 0$

$a_0(2(r^2-r) - r + 1) = 0$

$a_0(2r^2 - 2r - r + 1) = 0$

$a_0(2r^2 - 3r + 1) = 0$

Since we know $a_0 \neq 0$, we know that

$2r^2 - 3r + 1 = 0$. ← This is called the indicial eqn.

The soln is called the index or singularity exponent.

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{3 \pm \sqrt{9-8}}{4}$$

$$= \frac{3 \pm 1}{4}$$

$$= \frac{1}{2}, 1$$

$r_1 = 1$, $r_2 = \frac{1}{2}$ Note: By convention, if r_1 and r_2 are real roots, we enumerate the roots s.t. $r_1 > r_2$.
I.e. If we have 2 real roots, we let r_1 be the bigger root and r_2 be the smaller root.

Going back to the eqn. For $n \geq 1$, we have

$$2(n+r)(n+r-1)a_n - (n+r)a_n + a_n + a_{n-1} = 0$$

$$a_n\left(2(n+r)(n+r-1) - (n+r) + 1\right) + a_{n-1} = 0$$

$$a_n\left(2(n+r)^2 - 3(n+r) + 1\right) = -a_{n-1}$$

$$a_n = \frac{-a_{n-1}}{2(n+r)^2 - 3(n+r) + 1}$$

$$= \frac{-a_{n-1}}{\underbrace{(n+r-1)(2(n+r)-1)}}, \quad n = 1, 2, 3, \ldots$$

Note: The denominator may not always factor/simplify.

Take $r_1 = 1$. We will plug some values for n.

$$a_1 = \frac{-a_0}{1 \times 3} = \frac{-1}{1 \cdot 3} \leftarrow n=1$$

$n=2$:

$$a_2 = \frac{-a_1}{2 \times 5} = \frac{a_0}{1 \times 2 \times 3 \times 5} = \frac{1}{1 \cdot 2 \cdot 3 \cdot 5}$$

$n=3$:

$$a_3 = \frac{-a_2}{3 \times 7} = \frac{-a_0}{(1 \times 2 \times 3)(3 \times 5 \times 7)} = \frac{-1}{(1 \cdot 2 \cdot 3)(3 \cdot 5 \cdot 7)}$$

Note: $a_0 = 1$, always

Note: For finding a series soln about a singular point, we just need to calculate a_1, a_2, and a_3. We do not need to write/come up with a formula/summation.

Take $r=\frac{1}{2}$. We'll plug some values for n.

n=1:

$$a_1 = \frac{-a_0}{\frac{1}{2}\cdot 2} = -a_0 = -1$$

n=2:

$$a_2 = \frac{-a_1}{\frac{3}{2}\cdot 4} = \frac{-a_1}{2\cdot 3} = \frac{a_0}{2\cdot 3} = \frac{1}{2\cdot 3}$$

n=3:

$$a_3 = \frac{-a_2}{\frac{5}{2}\cdot 6} = \frac{-a_2}{3\cdot 5} = \frac{-1}{(2\cdot 3)(3\cdot 5)}$$

Recall that

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Expanding the summation, we get

$$y = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \ldots$$

For $r_1 = 1$:

$$y_1 = x - \frac{x^2}{3} + \frac{x^3}{30} - \frac{x^4}{630} + \cdots$$

For $r_2 = \frac{1}{2}$:

$$y_2 = x^{1/2} - x^{3/2} + \frac{x^{5/2}}{6} - \frac{x^{7/2}}{90} + \dots$$

$$y = c_1 y_1 + c_2 y_2$$

- There are 3 cases we have deal with:

1. $r_1, r_2 \in R$ and $r_1 - r_2 \neq n \in Z$ and $r_1 \neq r_2$
2. $r_1 = r_2$, and $r_1, r_2 \in R$
3. $r_1 - r_2 = n \in Z$ and $r_1, r_2 \in R$

Case 1:

The first case occurs when r_1 and r_2 are real numbers, and they don't equal to each other and their difference is not an integer. Example 4 is an example of case 1.

E.g. 5 Find a series soln for $2xy'' + y' + xy = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

$2xy'' + y' + xy = 0$ becomes

$$\sum_{n=0}^{\infty} 2a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

Notice that we can't get x^{n+r} for the first 2 summations because if we add 1 to the function, then we have to subtract 1 from the starting value, which is 0. Hence, if we subtract 1 from 0, we get a starting value of -1. Instead, we can make the third summation also have x^{n+r-1}.

Now we have:

$$\sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} +$$

$$\sum_{n=2}^{\infty} a_{n-2}x^{n+r-1} = 0$$

Take $n=0$. We get:

$2a_0(r)(r-1) + a_0(r) = 0$

$a_0(2r(r-1)+r) = 0$

$a_0(2r^2-2r+r) = 0$

$a_0(2r^2-r) = 0$

Recall: $a_0 \neq 0$

Hence, $2r^2 - r = 0$

$r(2r-1) = 0$

$r_1 = \frac{1}{2}$, $r_2 = 0$

Notice that:

1. $r_1, r_2 \in \mathbb{R}$
2. $r_1 \neq r_2$
3. $r_1 - r_2 = \frac{1}{2} \notin \mathbb{Z}$

Take $n \geq 2$:

$$2a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} = 0$$

$$a_n(2(n+r)(n+r-1) + (n+r)) = -a_{n-2}$$

$$a_n = \frac{-a_{n-2}}{(n+r)(2(n+r)-1)}$$

Take $r = \frac{1}{2}$. We'll plug some values for n:

$n=2$:

$$a_2 = \frac{-a_0}{(\frac{5}{2})(4)} = \frac{-1}{2 \cdot 5}$$

$n=4$:

$$a_4 = \frac{-a_2}{(\frac{9}{2})(8)} = \frac{1}{2 \cdot 4 \cdot 5 \cdot 9}$$

$n=6$:

$$a_6 = \frac{-a_4}{(\frac{13}{2})(12)} = \frac{-1}{2 \cdot 4 \cdot 5 \cdot 6 \cdot 9 \cdot 13}$$

Note: The reason why we skipped a_3 and a_5 is because we don't know what a_1 is.

Take $r=0$. We'll plug some values for n:

$n=2$:

$$a_2 = \frac{-a_0}{2 \cdot 3} = \frac{-1}{2 \cdot 3}$$

$n=4$:

$$a_4 = \frac{-a_2}{4\cdot 7} = \frac{1}{2\cdot 3\cdot 4\cdot 7}$$

$n=6$:

$$a_6 = \frac{-a_4}{6\cdot 11} = \frac{-1}{2\cdot 3\cdot 4\cdot 6\cdot 7\cdot 11}$$

Recall that $y = \sum_{n=0}^{\infty} a_n x^{n+r}$.

$$y_1 = x^{1/2} - \frac{x^2\cdot x^{1/2}}{2\cdot 5} + \frac{x^4\cdot x^{1/2}}{2\cdot 4\cdot 5\cdot 9} - \dots$$

$$y_2 = 1 - \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4\cdot 7} - \dots$$

$$y = c_1 y_1 + c_2 y_2$$

E.g. 6 Find a series soln for $4xy'' + 2y' + y = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1},$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $4xy'' + 2y' + y = 0$ as

$$4\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} + 2\sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} +$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

To get all 3 summations to have x^{n+r-1}, we'll change the last summation.

$$\sum_{n=0}^{\infty} a_n x^{n+r} \Leftrightarrow \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Now, we have

$$4\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} + 2\sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} +$$

$$\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

Take $n=0$. We get:

$$4a_0(r)(r-1)+2a_0(r)=0$$
$$a_0(4r(r-1)+2r)=0$$
$$a_0(4r^2-4r+2r)=0$$
$$2a_0(2r^2-2r+r)=0$$
$$a_0(2r^2-r)=0$$
$$2r^2-r=0$$
$$r(2r-1)=0$$
$$r_1=\tfrac{1}{2},\ r_2=0$$

Take $n \geq 1$:

$$4a_n(n+r)(n+r-1)+2a_n(n+r)+a_{n-1}=0$$
$$2a_n\left[2(n+r)(n+r-1)+(n+r)\right]=-a_{n-1}$$
$$a_n\left[(n+r)(2(n+r)-1)\right]=\frac{-a_{n-1}}{2}$$

$$a_n=\frac{-a_{n-1}}{2(n+r)(2(n+r)-1)}$$

Take $r=\frac{1}{2}$. We'll plug some values for n.

$n=1$:
$$a_1=\frac{-a_0}{2(\frac{3}{2})(2)}=\frac{-1}{2\cdot 3}$$

$n=2$:
$$a_2=\frac{-a_1}{2(\frac{5}{2})(4)}=\frac{1}{2\cdot 3\cdot 4\cdot 5}$$

$n=3$:
$$a_3=\frac{-a_2}{2(\frac{7}{2})(6)}=\frac{-1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$$

Take $r=0$. We'll plug some values for n.

$n=1$:

$$a_1 = \frac{-a_0}{2} = \frac{-1}{2}$$

$n=2$:

$$a_2 = \frac{-a_1}{4 \cdot 3} = \frac{1}{2 \cdot 3 \cdot 4}$$

$n=3$:

$$a_3 = \frac{-a_2}{6 \cdot 5} = \frac{-1}{6!}$$

Recall that $y = \sum_{n=0}^{\infty} a_n x^{n+r}$.

Expanding the summation, we get

$y = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \ldots$

Take $r = \frac{1}{2}$

$$y_1 = x^{1/2} - \frac{x^{3/2}}{3!} + \frac{x^{5/2}}{5!} - \frac{x^{7/2}}{7!} + \ldots$$

Take $r=0$

$$y_2 = 1 - \frac{x}{2!} + \frac{x^2}{3!} - \ldots$$

$y = c_1 y_1 + c_2 y_2$

Case 2:

The second case occurs when we have repeated roots. We will have to use Frobenius Method to find the second soln.

E.g. 7. Find a series soln for $xy'' + y' - y = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1},$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $xy'' + y' - y = 0$ as

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} -$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

We can change the third summation so that it has x^{n+r-1}.

$$\sum_{n=0}^{\infty} a_n x^{n+r} \Longleftrightarrow \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Now, we have

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} -$$

$$\sum_{n=1}^{\infty} a_{n-1}x^{n+r-1} = 0$$

Take $n=0$:

$a_0\left[r(r-1)+r\right]=0$

$r^2-r+r=0$

$r^2=0$

$r_1=r_2=0 \leftarrow$ Repeated Roots

Take $n\geq 1$:

$a_n(n+r)(n+r-1)+a_n(n+r)-a_{n-1}=0$

$a_n\left[(n+r)(n+r-1)+(n+r)\right]=a_{n-1}$

$$a_n = \frac{a_{n-1}}{(n+r)^2}$$

Take $r=0$. We'll plug some values for n.

$n=1$:

$$a_1 = \frac{a_0}{1^2} = 1$$

$n=2$:

$$a_2 = \frac{a_1}{2^2} = \frac{1}{4}$$

$n=3$:

$$a_3 = \frac{a_2}{3^2} = \frac{1}{1^2 \cdot 2^2 \cdot 3^2} = \frac{1}{(3!)^2}$$

$$a_n = \frac{1}{(n!)^2}$$

Recall that $y = \sum_{n=0}^{\infty} a_n x^{n+r}$.

Expanding the sum, we get
$y = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \dots$

$$y_1 = 1 + x + \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} + \dots$$

Now, we will use Frobenius Method to find y_2.

First, let's go back to the recurrence eqn.

$$a_n(r) = \frac{a_{n-1}(r)}{(n+r)^2}$$

$$y_1(r,x) = \sum_{n=0}^{\infty} a_n(r) x^{n+r}$$

Now, $y_2(x) = \partial_r y_1(r,x)|_{r=r_1}$

$$\partial_r y_1(r,x) = \partial_r \sum_{n=0}^{\infty} a_n(r) x^{n+r}$$

$$= \sum_{n=0}^{\infty} a'_n(r) x^{n+r} + \sum_{n=0}^{\infty} a_n(r) \partial_r (x^{n+r})$$

$$= \sum_{n=0}^{\infty} a'_n(r) x^{n+r} + \log(x) \sum_{n=0}^{\infty} a_n(r) x^{n+r}$$

$$= \log(x)\, y_1 + \sum_{n=0}^{\infty} a'_n(r) x^{n+r}$$

$$y_2 = \log(x)\, y_1 + \sum_{n=0}^{\infty} a'_n(r) x^{n+r} \Big|_{r=r_1}$$

Let's find a few terms for y_2.

$$a'_1 = \partial_r (a_1(r))$$
$$= \partial_r \left(\frac{1}{(n+r)^2} \right)$$
$$= \partial_r \left(\frac{1}{(1+r)^2} \right)$$
$$= \frac{-2}{(1+r)^3}$$

$$a'_1 x^{n+r} \Big|_{\substack{n=1 \\ r=0}} \rightarrow -2x$$

$$a_2' = \partial r\left(\frac{1}{(1+r)^2(2+r)^2}\right)$$

$$= \partial r\left(\frac{1}{((1+r)(2+r))^2}\right)$$

$$= \partial r\left(\frac{1}{(r^2+3r+2)^2}\right)$$

$$= \frac{-2(r^2+3r+2)(2r+3)}{(r^2+3r+2)^4}$$

$$= \frac{-2(2r+3)}{(r^2+3r+2)^3}$$

$$a_2' x^{n+r}\Big|_{\substack{n=2\\r=0}} \longrightarrow \frac{-2(3)}{2^3}x^2$$

$$= \frac{-6}{8}x^2$$

$$= \frac{-3}{4}x^2$$

$$y_2 = \log x \cdot y_1 - 2x - \tfrac{3}{4}x^2 + \ldots$$

E.g. 8 Find a series soln to $xy'' + y' + xy = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1},$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $xy'' + y' + xy = 0$ as

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} +$$

$$\sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

which is equivalent to

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} +$$

$$\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0$$

Take $n=0$:

$a_0 (r(r-1) + r) = 0$

$r^2 - r + r = 0$

$r = 0 \Rightarrow r_1 = r_2 = 0$

Take $n \geq 2$:

$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} = 0$

$a_n((n+r)(n+r-1) + (n+r)) = -a_{n-2}$

$$a_n = \frac{-a_{n-2}}{(n+r)^2}$$

Take $r=0$. Let's plug some values for n.

$n=2$:

$$a_2 = \frac{-a_0}{2^2} = \frac{-1}{2^2}$$

$n=4$:

$$a_4 = \frac{-a_2}{4^2} = \frac{1}{4^2 \cdot 2^2}$$

$n=6$:

$$a_6 = \frac{-a_4}{6^2} = \frac{-1}{2^2 \cdot 4^2 \cdot 6^2}$$

$$\begin{aligned} y_1 &= a_0 x^r + a_2 x^{r+2} + a_4 x^{r+4} + a_6 x^{r+6} \\ &= 1 - \frac{x^2}{4} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots \end{aligned}$$

Now, let's find Y_2.

$$y_2 = \log(x)\, y_1 + \sum_{n=0}^{\infty} a_n'^{(r)} x^{n+r} \Big|_{r=r_1}$$

$$a_2' = \partial r \left(\frac{1}{(2+r)^2} \right)$$

$$= \frac{-2}{(2+r)^3}$$

$$a_2' x^{2+r} \Big|_{r=0} \rightarrow \frac{-2}{8} x^2$$

$$= \frac{-x^2}{4}$$

$$a_4' = \partial r \left(\frac{1}{(4+r)^2 (2+r)^2} \right)$$

$$= \partial r \left(\frac{1}{((4+r)(2+r))^2} \right)$$

$$= \partial r \left(\frac{1}{(r^2+6r+8)^2} \right)$$

$$= \frac{-2(r^2+6r+8)(2r+6)}{(r^2+6r+8)^4}$$

$$a_4' x^{4+r} \Big|_{r=0} \rightarrow \frac{-2(6)}{8^3} x^4$$

$$= \frac{-12}{512} x^4$$

$$= \frac{-3x^4}{128}$$

Case 3:

The third and final case occurs when $r_1 - r_2$ is an integer, greater than 0. Here, we will use the Frobenius Method again. The Frobenius Theorem states that there is always a linearly independent soln

$$y_2 = a\log(x)\,y_1 + x^{r_2}\left(1 + \sum_{n=1}^{\infty} C_n(r_2)x^n\right) \quad \text{where}$$

$$a = \lim_{r \to r_2} (r - r_2)\,a_N(r), \quad N = R_1 - R_2$$

$$C_n = \left[(r - r_2)\,a_n(r)\right]'\Big|_{r=r_2}$$

Note: a could be 0.

E.g. 9 Find a series soln to $xy'' - y = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $xy'' - y = 0$ as

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\rightarrow \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Take $n=0$:

$a_0(r)(r-1)=0$

$r_1=1,\ r_2=0$

Notice that $r_1-r_2=1$, a positive integer.

Take $n\geq 1$:

$$a_n = \frac{a_{n-1}}{(n+r)(n+r-1)}$$

Take $r=1$. Let's plug some values for n.

$n=1$:

$$a_1 = \frac{a_0}{(2)(1)} = \frac{1}{2}$$

$n=2$:

$$a_2 = \frac{a_1}{(3)(2)} = \frac{1}{12}$$

$n=3$:

$$a_3 = \frac{a_2}{4 \cdot 3} = \frac{1}{144}$$

Recall that $y=\sum_{n=0}^{\infty} a_n x^{n+r}$.

Expanding the sum, we get

$y=a_0x^r + a_1x^{r+1} + a_2x^{r+2} + \ldots$

$$y_1 = x + \frac{x^2}{2} + \frac{x^3}{12} + \frac{x^4}{144} + \ldots$$

$$y_2 = a \log x \, y_1 + x^{r_2} \left[1 + \sum_{n=1}^{\infty} c_n(r_2) x^n \right]$$

$$a = \lim_{r \to r_2} (r - r_2) a_N(r)$$

$$= \lim_{r \to 0} r a_1(r)$$ **Note:** We have a_1 because $r_1 - r_2 = 1$.

$$= \lim_{r \to 0} r \left(\frac{a_0}{r(r+1)} \right)$$

$$= \lim_{r \to 0} \frac{1}{r+1}$$

$$= 1$$

$$c_n = \left[(r - r_2) a_n(r) \right]' \Big|_{r=r_2}$$

$$c_1 = \left[(r - r_2) a_1(r) \right]' \Big|_{r=r_2}$$

$$= \left(\frac{r \, a_0}{r(r+1)} \right)' \Big|_{r=r_2}$$

$$= \left(\frac{1}{r+1} \right)' \Big|_{r=r_2}$$

$$= \frac{-1}{(r+1)^2} \Big|_{r=0}$$

$$= -1$$

$$C_2 = \left[(r-r_2)a_2(r)\right]'\Big|_{r=r_2}$$
$$= \left(\frac{r\,a_1}{(r+1)(r+2)}\right)'\Big|_{r=r_2}$$
$$= \left(\frac{r\,a_0}{r(r+1)^2(r+2)}\right)'\Big|_{r=r_2}$$
$$= \left(\frac{1}{(r+1)^2(r+2)}\right)'\Big|_{r=r_2}$$
$$= -\frac{5}{4}$$

$$C_3 = \left[(r-r_2)a_3(r)\right]'\Big|_{r=r_2}$$
$$= \left(\frac{r\,a_2(r)}{(r+3)(r+2)}\right)'\Big|_{r=r_2}$$
$$= \left(\frac{r\,a_0(r)}{r(r+1)^2(r+2)^2(r+3)}\right)'\Big|_{r=r_2}$$
$$= \left(\frac{1}{(r+1)^2(r+2)^2(r+3)}\right)'\Big|_{r=r_2}$$
$$= -\frac{5}{18}$$

$$y_2 = \log(x)\,y_1 + 1 - x - \frac{5}{4}x^2 - \frac{5}{18}x^3 \cdots$$

More Examples:

E.g. 10 Find a series soln to $xy'' + y' - y = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1},$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $xy'' + y' - y = 0$ as

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} -$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

We can change the last summation so that it has x^{n+r-1}.

$$\sum_{n=0}^{\infty} a_n x^{n+r} \Longleftrightarrow \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Now, we have

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} -$$

$$\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Take $n=0$.

$a_0 \left(r(r-1) + r \right) = 0$

$r^2 - r + r = 0$

$r^2 = 0 \rightarrow r_1 = r_2 = 0$

Take $n \geq 1$

$a_n (n+r)(n+r-1) + a_n (n+r) - a_{n-1} = 0$

$a_n \left((n+r)(n+r-1) + (n+r) \right) = a_{n-1}$

$a_n \left((n+r)^2 \right) = a_{n-1}$

$$a_n = \frac{a_{n-1}}{(n+r)^2}$$

Take $r=0$. We'll plug some values for n.

$n=1$:

$$a_1 = \frac{a_0}{1^2} = 1$$

$n=2$:

$$a_2 = \frac{a_1}{2^2} = \frac{1}{4}$$

$n=3$:

$$a_3 = \frac{a_2}{3^2} = \frac{1}{9 \cdot 4} = \frac{1}{36}$$

Recall that $y = \sum_{n=0}^{\infty} a_n x^{n+r}$.

Expanding the sum, we get
$y = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \dots$

$$y_1 = 1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \dots$$

$$y_2 = \log(x)\, y_1 + \sum_{n=0}^{\infty} a_n'(r)\, x^{n+r} \Big|_{r=r_1}$$

$$\begin{aligned} a_1' &= \partial_r (a_1(r)) \\ &= \partial_r \left(\frac{1}{(n+r)^2} \right) \\ &= \partial_r \left(\frac{1}{(1+r)^2} \right) \\ &= \frac{-2}{(1+r)^3} \end{aligned}$$

$$a_1' x^{n+r} \Big|_{\substack{n=1 \\ r=0}} \longrightarrow -2x$$

$$\begin{aligned} a_2' &= \partial_r (a_2(r)) \\ &= \partial_r \left(\frac{1}{(1+r)^2 (2+r)^2} \right) \\ &= \partial_r \left(\frac{1}{(r^2+3r+2)^2} \right) \\ &= \frac{-2(r^2+3r+2)(2r+3)}{(r^2+3r+2)^4} \\ &= \frac{-2(2r+3)}{(r^2+3r+2)^3} \end{aligned}$$

$$a_2' x^{n+r} \Big|_{\substack{n=2 \\ r=0}} \Rightarrow -\frac{3}{4}x^2$$

$$y_2 = \log(x)\, y_1 - 2x - \frac{3}{4}x^2 + \dots$$

E.g. 11 Find a series soln to $xy'' + y = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

$$xy'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1}$$

We can rewrite $xy'' + y = 0$ as

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

Take $n=0$.

$a_0 (r)(r-1) = 0$

$r_1 = 1, \; r_2 = 0$

Take $n=1$.

$a_n (n+r)(n+r-1) + a_{n-1} = 0$

$$a_n = \frac{-a_{n-1}}{(n+r)(n+r-1)}$$

Take $r=1$. We'll plug some values for n.

$n=1$:

$$a_1 = \frac{-a_0}{2} = \frac{-1}{2}$$

$n=2$:

$$a_2 = \frac{-a_1}{(3)(2)} = \frac{1}{3\cdot 2\cdot 2} = \frac{1}{12}$$

$n=3$:

$$a_3 = \frac{-a_2}{4\cdot 3} = \frac{-1}{144}$$

$$y_1 = x - \frac{x^2}{2} + \frac{x^3}{12} - \frac{x^4}{144}$$

$$y_2 = a \log x \, y_1 + x^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n(r_2) x^n\right)$$

$$a = \lim_{r \to 0} (r-0) a_1(r)$$

$$= \lim_{r \to 0} \left(\frac{-r}{r(r+1)}\right)$$

$$= \lim_{r \to 0} \left(\frac{-1}{r+1}\right)$$

$$= -1$$

$$C_1 = [(r-r_2)a_1(r)]'\Big|_{r=r_2}$$
$$= \left(\frac{r(-1)}{(r)(r+1)}\right)'\Big|_{r=r_2}$$
$$= \left(\frac{-1}{r+1}\right)'\Big|_{r=r_2}$$
$$= 1$$

$$C_2 = [(r-r_2)a_2(r)]'\Big|_{r=r_2}$$
$$= \left(\frac{r(1)}{(r+2)(r+1)^2 r}\right)'\Big|_{r=r_2}$$
$$= \left(\frac{1}{(r+2)(r+1)^2}\right)'\Big|_{r=r_2}$$
$$= \left(\frac{1}{x^3+4x^2+5x+2}\right)'\Big|_{r=r_2}$$
$$= \frac{-5}{4}$$

$$y_2 = -\log x\, y_1 + 1 + x - \frac{5x^2}{4} + \ldots$$